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Extended Bell and Stirling Numbers From Hypergeometric Exponentiation

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Abstract

Exponentiating the hypergeometric series \( _0F_L(1,1,\ldots,1;z) \), \( L = 0,1,2,\ldots \), furnishes a recursion relation for the members of certain integer sequences \( b_L(n) \), \( n = 0,1,2,\ldots \). For \( L > 0 \), the \( b_L(n) \)'s are generalizations of the conventional Bell numbers, \( b_0(n) \). The corresponding associated Stirling numbers of the second kind are also investigated. For \( L = 1 \) one can give a combinatorial interpretation of the numbers \( b_1(n) \) and of some Stirling numbers associated with them. We also consider the \( L \geq 1 \) analogues of Bell numbers for restricted partitions.

The conventional Bell numbers \( b_0(n) \), \( n = 0,1,2,\ldots \), have a well-known exponential generating function

\[
B_0(z) \equiv e^{(e^z - 1)} = \sum_{n=0}^{\infty} b_0(n) \frac{z^n}{n!},
\]

which can be derived by interpreting \( b_0(n) \) as the number of partitions of a set of \( n \) distinct elements. In this note we obtain recursion relations for related sequences of positive integers, called \( b_L(n) \), \( L = 0,1,2,\ldots \),
obtained by exponentiating the hypergeometric series \( _0F_L(1,1,\ldots,1; z) \) defined by \[ (2) \]:

\[
_0F_L(1,1,\ldots,1; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{L+1}},
\]

which we shall denote by \( _0F_L(z) \) and which includes the special cases \( _0F_0(z) \equiv e^z \) and \( _0F_1(z) \equiv I_0(2\sqrt{z}) \), where \( I_0(x) \) is the modified Bessel function of the first kind. For \( L > 1 \), the functions \( _0F_L(z) \) are related to the so-called hyper-Bessel functions \([3],[4],[5]\), which have recently found application in quantum mechanics \([6],[7]\). Thus we are interested in \( b_L(n) \) given by

\[
e^{[0F_L(z)-1]} = \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}},
\]

thereby defining a hypergeometric generating function for the numbers \( b_L(n) \). From eq. \([8]\) it follows formally that

\[
b_L(n) = (n!)^L \cdot \left. \frac{d^n}{dz^n} \left(e^{[0F_L(z)-1]}\right)\right|_{z=0}.
\]

For \( L = 0 \) the r.h.s of eq. \([8]\) can be evaluated in closed form:

\[
b_0(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} = \left\{ \frac{1}{e^z} \left[ \left( \frac{d}{dz} \right)^n e^z \right] \right\}_{z=1}.
\]

The first equality in \([9]\) is the celebrated Dobiński formula \([9],[10],[11]\). The second equality in eq. \([8]\) follows from observing that for a power series \( R(z) = \sum_{k=0}^{\infty} A_k z^k \) we have

\[
\left( \frac{z}{d/dz} \right)^n R(z) = \sum_{k=0}^{\infty} A_k k^n z^k
\]

and applying eq. \([8]\) to the exponential series \( (A_k = (k!)^{-1}) \).

The reason for including the divisors \((n!)^{L+1}\) rather than \(n!\) as in the usual exponential generating function arises from the fact that only by using eq. \([12]\) are the numbers \( b_L(n) \) actually integers. This can be seen from general formulas for exponentiation of a power series \([13]\), which employ the (exponential) Bell polynomials, complicated and rather unwieldy objects. It cannot however be considered as a proof that the \( b_L(n) \) are integers. At this stage we shall use eq. \([12]\) with \( b_L(n) \) real and apply to it an efficient method, described in \([14]\), which will yield the recursion relation for the \( b_L(n) \). (For the proof that the \( b_L(n) \) are integers, see below eq. \([16]\)). To this end we first obtain a result for the multiplication of two power-series of the type \([12]\). Suppose we wish to multiply \( f(x) = \sum_{n=0}^{\infty} a_L(n) \frac{x^n}{(n!)^{L+1}} \) and \( g(x) = \sum_{n=0}^{\infty} c_L(n) \frac{x^n}{(n!)^{L+1}} \). We get

\[
f(x) \cdot g(x) = \sum_{n=0}^{\infty} d_L(n) \frac{x^n}{(n!)^{L+1}},
\]

where

\[
d_L(n) = (n!)^{L+1} \sum_{r+s=n} \frac{a_L(r) c_L(s)}{(r!)^{L+1}(s!)^{L+1}} = \sum_{r=0}^{\infty} \binom{n}{r} L+1 \cdot a_L(r) c_L(n-r).
\]

Substitute eq. \([12]\) into eq. \([16]\) and take the logarithm of both sides of eq. \([16]\):

\[
\sum_{n=1}^{\infty} \frac{z^n}{(n!)^{L+1}} = \ln \left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right).
\]
Now differentiate both sides of eq. (8) and multiply by $z$:

\[
\left( \sum_{n=0}^{\infty} b_L(n) \frac{z^n}{(n!)^{L+1}} \right) \left( \sum_{n=0}^{\infty} n \frac{z^n}{(n!)^{L+1}} \right) = \sum_{n=0}^{\infty} n b_L(n) \frac{z^n}{(n!)^{L+1}},
\]

which with eq. (1) yields the desired recurrence relation

\[
b_L(n + 1) = \frac{1}{n + 1} \sum_{k=0}^{n} \binom{n+1}{k} b_L(k), \quad n = 0, 1, \ldots \tag{10}
\]

\[
b_L(0) = 1. \tag{12}
\]

Since eq. (11) involves only positive integers, it follows that the $b_L(n)$ are indeed positive integers. For $L = 0$ one gets the known recurrence relation for the Bell numbers \( b_0(z) \):

\[
b_0(n + 1) = \sum_{k=0}^{n} \binom{n}{k} b_0(k). \tag{13}
\]

We have used eq. (1) to calculate some of the $b_L(n)$’s, listed in Table I, for $L = 0, 1, \ldots, 6$. Eq. (1) for $n$ fixed, gives closed form expressions for the $b_L(n)$ directly as a function of $L$ (columns in Table I):

- $b_L(2) = 1 + 2^L$,
- $b_L(3) = 1 + 3 \cdot 3^L + (3!)^L$,
- $b_L(4) = 1 + 4 \cdot 4^L + 3 \cdot 6^L + 6 \cdot 12^L + (4!)^L$, etc.

The sets of $b_L(n)$ have been checked against the most complete source of integer sequences available \( [10] \). Apart from the case $L = 0$ (conventional Bell numbers) only the first non-trivial sequence $L = 1$ is listed\( [10] \); it turns out that this sequence $b_1(n)$, listed under the heading A023998 in \( [10] \), can be given a combinatorial interpretation as the number of block permutations on a set of $n$ objects which are uniform, i.e. corresponding blocks have the same size $[12]$. \( ^1 \)

Eq. (1) can be generalized by including an additional variable $x$, which will result in “smearing out” the conventional Bell numbers $b_0(n)$ with a set of integers $S_0(n, k)$, such that for $k > n$, $S_0(n, k) = 0$, and $S_0(0, 0) = 1$, $S_0(n, 0) = 0$. In particular,

\[
B_0(z, x) \equiv e^{x(e^z - 1)} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_0(n, k) x^k \right] \frac{z^n}{n!}, \tag{14}
\]

which leads to the (exponential) generating function of $S_0(n, l)$, the conventional Stirling numbers of the second kind, (see [1], [8]), in the form

\[
\frac{(e^z - 1)^l}{l!} = \sum_{n=l}^{\infty} \frac{S_0(n, l)}{n!} \frac{z^n}{n!}, \tag{15}
\]

and defines the so-called exponential or Touchard polynomials $l_n^{(0)}(x)$ as

\[
l_n^{(0)}(x) = \sum_{k=1}^{n} S_0(n, k)x^k. \tag{16}
\]

They satisfy

\[
l_n^{(0)}(1) = b_0(n), \tag{17}
\]

\( ^1 \) (others have since been added)

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justifying the term “smearing out” used above.

The appearance of integers in eq. (3) suggests a natural extension with an additional variable $x$:

$$B_L(z, x) \equiv e^{x[F_L(z) - 1]} = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} S_L(n, k) x^k \right] \frac{z^n}{(n!)^{L+1}}, \quad (18)$$

where we include the right divisors $(n!)^{L+1}$ in the r.h.s of (18).

This in turn defines “hypergeometric” polynomials of type $L$ and order $n$ through

$$l_n^{(L)}(x) = \sum_{k=1}^{n} S_L(n, k) x^k, \quad (19)$$

which satisfy

$$l_n^{(L)}(1) = b_L(n), \quad (20)$$

with the $b_L(n)$ of eq. (14). Thus the polynomials of eq. (14) “smear out” the $b_L(n)$ with the generalized Stirling numbers of the second kind, of type $L$, denoted by $S_L(n, k)$ (with $S_L(n, k) = 0$, if $k > n$, $S_L(n, 0) = 0$ if $n > 0$ and $S_L(0, 0) = 1$), which have, from eq. (13) the “hypergeometric” generating function

$$\frac{(_0F_L(z) - 1)^l}{l!} = \sum_{n=0}^{\infty} \frac{S_L(n, l)}{(n!)^{L+1}} z^n, \quad L = 0, 1, 2, \ldots \quad (21)$$

Eq. (21) can be used to derive a recursion relation for the numbers $S_L(n, k)$, in the same manner as eq. (3) yielded eq. (4). Thus we take the logarithm of both sides of eq. (21), differentiate with respect to $z$, multiply by $z$ and obtain:

$$\left( \sum_{n=0}^{\infty} \frac{S_L(n, l - 1)}{(n!)^{L+1}} z^n \right) \left( \sum_{n=0}^{\infty} \frac{n}{(n!)^{L+1}} z^n \right) = \sum_{n=0}^{\infty} \frac{n S_L(n, l)}{(n!)^{L+1}} z^n, \quad (22)$$

which, with the help of eq. (3), produces the required recursion relation

$$S_L(n + 1, l) = \sum_{k=l-1}^{n} \binom{n}{k} \binom{n + 1}{k}^L S_L(k, l - 1), \quad (23)$$

$$S_L(0, 0) = 1, \quad S_L(n, 0) = 0, \quad (24)$$

which for $L = 0$ is the recursion relation for the conventional Stirling numbers of the second kind [1, 8], and in eq. (24) the appropriate summation range has been inserted. Since the recursions of eq. (23) and eq. (24) involve only integers we conclude that $S_L(n, l)$ are positive integers.

We have calculated some of the numbers $S_L(n, l)$ using eq. (21) and have listed them in Tables II and III, for $L = 1$ and $L = 2$ respectively. Observe that $S_1(n, 2) = \binom{2n + 1}{n + 1} - 1$ and $S_L(n, n) = (n!)^{L}, \quad L = 1, 2$.

Also, by fixing $n$ and $l$, the individual values of $S_L(n, l)$ have been calculated as a function of $L$ with the help of eq. (23), see Table IV, from which we observe

$$S_L(n, n) = (n!)^{L}, \quad L = 1, 2, \ldots \quad (25)$$

which is the lowest diagonal in Table IV. We now demonstrate that the repetitive use of eq. (23) permits one to establish closed-form expressions for any supra-diagonal of order $p$, i.e. the sequence $S_L(n + p, n)$,
for $p = 1, 2, 3, \ldots$, if one knows the expression for all $S_{L}(n + k, n)$ with $k < p$. We shall illustrate it here for $p = 1, 2$. To this end fix $l = n$ on both sides of eq. (23). It becomes, upon using eq. (25), and defining $\alpha_{L}(n) \equiv S_{L}(n + 1, n)$, a linear recursion relation

$$
\alpha_{L}(n) = \frac{n(n+1)!}{2^l} (n+1)^{L} \alpha_{L}(n-1), \quad \alpha_{L}(0) = 0,
$$

with the solution

$$
\alpha_{L}(n) = S_{L}(n+1, n) = \frac{n(n+1)}{2} \left[ \frac{(n+1)!}{2} \right]^{L}
$$

$$
= \left[ \frac{(n+1)!}{2} \right]^{L} S_{0}(n+1, n),
$$

which gives the second lowest diagonal in Table IV. Observe that for any $L$, $S_{L}(n+1, n)$ is proportional to $S_{0}(n+1, n) = n(n+1)/2$. The sequence $S_{1}(n+1, n) = 1, 9, 72, 600, 5400, 856480, \ldots$ is of particular interest: it represents the sum of inversion numbers of all permutations on $n$ letters [1]. For more information about this and related sequences see the entry A001809 in [1]. The $S_{L}(n+1, n)$ for $L > 1$ do not appear to have a simple combinatorial interpretation. A recurrence equation for $\beta_{L}(n) \equiv S_{L}(n+2, n)$ is obtained upon substituting eq. (25) and eq. (27) into eq. (23):

$$
\beta_{L}(n) = \frac{n(n+1)}{2!} \left[ \frac{(n+2)!}{2} \right]^{L} \left( \frac{n-1}{2^L} + \frac{1}{3^L} \right) + (n+2)^{L} \beta_{L}(n-1), \quad \beta_{L}(0) = 0.
$$

It has the solution

$$
S_{L}(n+2, n) = \frac{n(n+1)(n+2)}{3 \cdot 2^L} \left[ \frac{(n+2)!}{2} \right]^{L} \left( \frac{3}{2^L}(n-1) + \frac{4}{3^L} \right)
$$

which is a closed form expression for the second lowest diagonal in Table IV. Clearly, eq. (30) for $L = 0$ gives the combinatorial form for the series of conventional Stirling numbers

$$
S_{0}(n+2, n) = \frac{n(n+1)(n+2)(3n+1)}{4!}.
$$

In a similar way we obtain

$$
S_{L}(n+3, n) = \frac{n(n+1)(n+2)(n+3)}{3 \cdot 2^L} \left[ \frac{(n+3)!}{3} \right]^{L} \n\times \left( \frac{1}{8^L} + n \left( \frac{1}{4^L-1} - \frac{3^{L+1}}{8^L} \right) + \frac{2+2 \cdot 3^L}{8^L} - \frac{1}{4^L-1} \right)
$$

which for $L = 0$ reduces to

$$
S_{0}(n+3, n) = \frac{1}{48} n^2 (n+1)^2(n+2)(n+3).
$$

Combined with the standard definition [8], [9]

$$
S_{0}(n, l) = \frac{(-1)^l}{l!} \sum_{k=1}^{l} \frac{(-1)^{k}}{k} \left( \begin{array}{c} l \\ k \end{array} \right) k^n.
$$
eqs. (28), (31) and (33) give compact expressions for the summation form of $S_0(n+p,n)$. Further, from eq. (34), use of eq. (6) gives the following generating formula

$$S_0(n,l) = (-1)^l l! \left[ \left( zd/dz \right)^n \sum_{k=1}^{l} (-1)^k \binom{l}{k} z^k \right]_{z=1}, \quad n \geq l. \quad (35)$$

The formula (1) can be generalized by putting restrictions on the type of resulting partitions. The generating function for the number of partitions of a set of $n$ distinct elements without singleton blocks $b_0(1,n)$ is $[8], [14], [15], B_0(1,n) = e^{z-1-z} \sum_{n=0}^\infty b_0(1,n) \frac{z^n}{n!}, \quad (37)$

or more generally, without singleton, doubleton . . . , $p$-blocks ($p = 0, 1, \ldots$) is $[13]

$$B_0(p,z) = e^{z-\sum_{k=0}^p \frac{z^k}{k+1}} = \sum_{n=0}^\infty b_0(p,n) \frac{z^n}{n!}, \quad (38)$$

with the corresponding associated Stirling numbers defined by analogy with eq. (14) and eq. (22). The numbers $b_0(1,n)$, $b_0(2,n)$, $b_0(3,n)$, $b_0(4,n)$ can be read off from the sequences A000296, A006505, A057837 and A057814 in [10], respectively. For more properties of these numbers see [11].

We carry over this type of extension to eq. (11) and define $b_L(p,n)$ through

$$B_L(p,z) = e^{F_L(z)-\sum_{k=0}^p \frac{z^k}{(n+1)^{k+1}}} = \sum_{n=0}^\infty b_L(p,n) \frac{z^n}{(n!)^{L+1}}, \quad (39)$$

where $b_L(0,n) = b_L(n)$ from eq. (11). (We know of no combinatorial meaning of $b_L(p,n)$ for $L \geq 1$, $p > 0$).

The $b_L(p,n)$ satisfy the following recursion relations:

$$b_L(p,n) = \sum_{k=0}^{n-p} \binom{n}{k} \left( \frac{n+1}{k} \right)^L b_L(p,k), \quad (40)$$

$$b_L(p,0) = 1, \quad (41)$$

$$b_L(p,1) = b_L(p,2) = \cdots = b_L(p,p) = 0, \quad (42)$$

$$b_L(p,p+1) = 1. \quad (43)$$

That the $b_L(p,n)$ are integers follows from eq. (40). Through eq. (39) additional families of integer Stirling-like numbers $S_{L,p}(n,k)$ can be readily defined and investigated.

The numbers $b_0(p,n)$ are collected in Table V, and Tables VI and VII contain the lowest values of $b_1(p,n)$ and $b_2(p,n)$, respectively.

Formula (11) can be used to express $e$ in terms of $b_0(n)$ in various ways. Two such lowest order (in differentiation) forms are

$$e = 1 + \ln \left( \sum_{n=0}^\infty \frac{b_0(n)}{n!} \right) = \ln \left( \sum_{n=0}^\infty \frac{b_0(n+1)}{n!} \right). \quad (44)$$

$$e = 1 + \ln \left( \sum_{n=0}^\infty \frac{b_0(n)}{n!} \right) = \ln \left( \sum_{n=0}^\infty \frac{b_0(n+1)}{n!} \right). \quad (45)$$
In the very same way, eq. (3) can be used to express the values of \( _0F_L(z) \) and its derivatives at \( z = 1 \) in terms of certain series of \( b_L(n) \)'s. For \( L = 1 \), the analogues of eq. (3) and eq. (4) are

\[
I_0(2) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n)}{(n!)^2} \right),
\]

\[
I_0(2) + \ln(I_1(2)) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_1(n+1)}{(n+1)(n!)^2} \right)
\]

and for \( L = 2 \) the corresponding formulas are

\[
_0F_2(1, 1; 1) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n)}{(n!)^3} \right),
\]

\[
_0F_2(1, 1; 1) + \ln ( _0F_2(2, 2; 1) ) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{b_2(n+1)}{(n+1)^2(n!)^3} \right).
\]

By fixing \( z_0 \) at values other than \( z_0 = 1 \), one can link the numerical values of certain combinations of \( _0F_L(1, \ldots; z_0) \), \( _0F_L(2, \ldots; z_0) \), \ldots and their logarithms, with other series containing the \( b_L(n) \)'s.

The above considerations can be extended to the exponentiation of the more general hypergeometric functions of type \( _0F_L(k_1, k_2, \ldots, k_L; z) \) where \( k_1, k_2, \ldots, k_L \) are positive integers. We conjecture that for every set of \( k_n \)'s a different set of integers will be generated through an appropriate adaptation of eq. (3).

We quote one simple example of such a series. For

\[
_0F_2(1, 2; z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)(n!)^3}
\]

eq. (3) extends to

\[
e^{[ _0F_2(1, 2; z) - 1]} = \sum_{n=0}^{\infty} \frac{f_2(n) z^n}{(n+1)(n!)^3}
\]

where the numbers

\[
f_2(n) = (n+1)(n!)^2 \left[ \frac{d^n}{dz^n} e^{[ _0F_2(1, 2; z) - 1]} \right]_{z=0}
\]

turn out to be integers: \( f_2(n), n = 0, 1, \ldots, 8 \) are: 1, 1, 4, 37, 641, 18276, 789377, 48681011, etc. (A061683).

The analogue of equations (23) and (44) is:

\[
_0F_2(1, 2; 1) = 1 + \ln \left( \sum_{n=0}^{\infty} \frac{f_2(n)}{(n+1)(n!)^3} \right).
\]

**Acknowledgements**

We thank L. Haddad for interesting discussions. We have used Maple\textsuperscript{©} to calculate most of the numbers discussed above.
Table I: Table of $b_L(n)$: for $L, n = 0, 1, \ldots, 6$. (The rows give sequences A000110, A023998, A061684–A061688.)

<table>
<thead>
<tr>
<th>$L$</th>
<th>$b_L(0)$</th>
<th>$b_L(1)$</th>
<th>$b_L(2)$</th>
<th>$b_L(3)$</th>
<th>$b_L(4)$</th>
<th>$b_L(5)$</th>
<th>$b_L(6)$</th>
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<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>3</td>
<td>16</td>
<td>131</td>
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<td>22 482</td>
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<td>69 026</td>
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<td>350 813 126</td>
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<td>3 464 129 078 126</td>
<td>173 566 857 025 139 312</td>
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</table>

Table II: Table of $S_L(n,l)$: for $L = 1$ and $l, n = 1, 2, \ldots, 8$. (The triangle, read by columns, gives A061691, the rows and diagonals give A017063, A061690, A000142, A001809, A061689.)

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<th>$S_1(1,l)$</th>
<th>$S_1(2,l)$</th>
<th>$S_1(3,l)$</th>
<th>$S_1(4,l)$</th>
<th>$S_1(5,l)$</th>
<th>$S_1(6,l)$</th>
<th>$S_1(7,l)$</th>
<th>$S_1(8,l)$</th>
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</tr>
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<td>24</td>
<td>600</td>
<td>10 500</td>
<td>161 700</td>
<td>2 361 016</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>5 400</td>
<td>161 700</td>
<td>4 116 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td>52 920</td>
<td>2 493 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5 040</td>
<td>564 480</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>40 320</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table III: Table of $S_L(n,l)$: for $L = 2$ and $l, n = 1, 2, \ldots, 8$. (The triangle, read by columns, gives A061692, the rows and diagonals give A061693, A061694, A001044, A061695.)

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_2(1,l)$</th>
<th>$S_2(2,l)$</th>
<th>$S_2(3,l)$</th>
<th>$S_2(4,l)$</th>
<th>$S_2(5,l)$</th>
<th>$S_2(6,l)$</th>
<th>$S_2(7,l)$</th>
<th>$S_2(8,l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>27</td>
<td>172</td>
<td>1 125</td>
<td>7 591</td>
<td>52 479</td>
<td>369 580</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>864</td>
<td>17 500</td>
<td>351 000</td>
<td>7 197 169</td>
<td>151 633 440</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>576</td>
<td>36 000</td>
<td>1 746 000</td>
<td>80 262 000</td>
<td>3 691 514 176</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14 400</td>
<td>1 944 000</td>
<td>191 394 000</td>
<td>17 188 416 000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>518 400</td>
<td>133 358 400</td>
<td>23 866 214 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>25 401 600</td>
<td>11 379 916 800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>1 625 702 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table IV: Table of $S_L(n, l)$: $l, n = 1, 2, \ldots, 6$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$S_L(1, l)$</th>
<th>$S_L(2, l)$</th>
<th>$S_L(3, l)$</th>
<th>$S_L(4, l)$</th>
<th>$S_L(5, l)$</th>
<th>$S_L(6, l)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$(2!)^L$</td>
<td>$3 \cdot 3^L$</td>
<td>$4 \cdot 4^L + 3 \cdot 6^L$</td>
<td>$5 \cdot 5^L + 10 \cdot 10^L$</td>
<td>$6 \cdot 6^L + 15 \cdot 15^L + 10 \cdot 20^L$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$(3!)^L$</td>
<td>$6 \cdot 12^L$</td>
<td>$10 \cdot 20^L + 15 \cdot 30^L$</td>
<td>$15 \cdot 30^L + 60 \cdot 60^L + 15 \cdot 90^L$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$(4!)^L$</td>
<td>$10 \cdot 60^L$</td>
<td>$20 \cdot 120^L + 45 \cdot 180^L$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$(5!)^L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table V: Table of $b_0(p, n)$: $p = 0, 1, 2, 3$; $n = 0, \ldots, 10$. (The columns give A000110, A000296, A006505, A057837.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_0(0, n)$</th>
<th>$b_0(1, n)$</th>
<th>$b_0(2, n)$</th>
<th>$b_0(3, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>52</td>
<td>11</td>
<td>1</td>
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</tr>
<tr>
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<td>203</td>
<td>41</td>
<td>11</td>
<td>1</td>
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<td>7</td>
<td>877</td>
<td>162</td>
<td>36</td>
<td>1</td>
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<tr>
<td>8</td>
<td>4140</td>
<td>715</td>
<td>92</td>
<td>36</td>
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<tr>
<td>9</td>
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<td>3425</td>
<td>491</td>
<td>127</td>
</tr>
<tr>
<td>10</td>
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<td>17722</td>
<td>2557</td>
<td>337</td>
</tr>
</tbody>
</table>

Table VI: Table of $b_1(p, n)$: $p = 0, 1, 2$; $n = 0, \ldots, 9$. (The columns give A023998, A061696, A061697.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_1(0, n)$</th>
<th>$b_1(1, n)$</th>
<th>$b_1(2, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
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<td>16</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>131</td>
<td>19</td>
<td>1</td>
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<td>5</td>
<td>1496</td>
<td>101</td>
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</tr>
<tr>
<td>6</td>
<td>22482</td>
<td>1776</td>
<td>201</td>
</tr>
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<td>7</td>
<td>426833</td>
<td>23717</td>
<td>1226</td>
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<td>8</td>
<td>9934563</td>
<td>515971</td>
<td>5587</td>
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<td>9</td>
<td>277006192</td>
<td>11893597</td>
<td>493333</td>
</tr>
</tbody>
</table>

9
### Table VII: Table of $b_2(p,n)$: $p = 0, 1, 2$; $n = 0, \ldots, 8$. (The columns give A061698–A061700.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_2(0,n)$</th>
<th>$b_2(1,n)$</th>
<th>$b_2(2,n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
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<td>4</td>
<td>1 613</td>
<td>109</td>
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</tr>
<tr>
<td>5</td>
<td>69 026</td>
<td>1 001</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>4 566 992</td>
<td>128 876</td>
<td>4 001</td>
</tr>
<tr>
<td>7</td>
<td>437 665 649</td>
<td>4 682 637</td>
<td>42 876</td>
</tr>
<tr>
<td>8</td>
<td>57 903 766 800</td>
<td>792 013 069</td>
<td>347 117</td>
</tr>
</tbody>
</table>

#### References


(Mentions sequences A000296 A001044 A001809 A006505 A010763 A023998 A057814 A057837 A061683 A061684 A061685 A061686 A061687 A061688 A061689 A061690 A061691 A061692 A061693 A061694 A061695 A061696 A061697 A061698 A061699 A061700.)

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